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On swimming in a visco-elastic liquid

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The velocity induced by a transversely waving infinite flexible sheet in a viscoelastic liquid is investigated by a method of successive approximation up to second order in the amplitude of oscillation of the sheet. The incompressible second-order fluid model has been used and it is found that the elastic property of the fluid augments self-propulsion (increases the induced velocity) in some range of Reynolds number (based on the phase velocity) and hampers it (reduces the induced velocity) in some other range with higher values of Reynolds number.

1. Introduction

In an analysis of the self-propulsion of microscopic organisms Taylor (1951) observed that 'though microscopic swimming creatures are certainly three-dimensional, yet the great simplicity of two-dimensional analysis makes it worth while to discuss the problem of self-propulsion in a viscous fluid in two dimensions', and considered the motion set up in an infinite fluid by a train of two-dimensional waves travelling across an inextensible flexible sheet. With the waving surface represented by

$$y = b\sin(\kappa x - \sigma t),$$

the wave propagating in the $+x$ direction with phase velocity $c = \sigma/\kappa$, Taylor took the field equation (neglecting inertia terms) $\nabla^4\psi = 0$, ψ being the two-dimensional stream function, and found that the oscillation of the waving surface induces a velocity

$$U_\infty = c[\tfrac{1}{2}(\kappa b)^2 + O(\kappa b)^4] \tag{1}$$

in the fluid at infinity in the $+x$ direction, so that, if the fluid is at rest far from the sheet, the sheet is propelled in the direction opposite to that of the propagation of the distorting wave. As the waves of infinitesimally small amplitude (for which the terms containing $b^2\kappa^2$ can be neglected) do not give rise to propulsive velocity, Taylor expanded the boundary conditions in powers of b to consider the effects of finite amplitude.

Taylor's analysis is limited to the case of vanishingly small Reynolds number R (based on the phase velocity of the wave). A generalization of Taylor's problem to include the effects of fluid inertia was first attempted by Reynolds (1965), who obtained a multiplicative correction factor (as a function of R) for propulsion velocity showing that the effect of fluid inertia is to increase the propulsion velocity for a particular wave amplitude. But later Tuck (1968) pointed out that Reynolds, in his analysis, erroneously anticipated the mean second-order flow to be purely uniform

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and effectively took account only of the first-order inertia terms involving $\partial/\partial t$, leaving aside the second-order convection terms. Tuck presented an interesting alternative derivation of the problem and gave the correct expression for U_∞. Tuck's result shows that fluid inertia in fact decreases the propulsion-velocity, reducing it to one-half of Taylor's value as $R \to \infty$. Taylor's inextensibility condition is not strictly necessary in Tuck's formulation and his analysis is valid for any predominantly transverse waving oscillation of a flexible sheet. The shape of the sheet need not be strictly sinusoidal in space or time, and the problem cannot be reduced exactly to a steady flow.

Blake (1971) studied two infinite-length models, (i) two-dimensional waving sheet, and (ii) axisymmetric waving cylinder, for ciliary propulsion of microscopic organisms. Like Taylor, he assumed creeping flow; but taking the surface to be extensible he considered longitudinal and transverse oscillations acting together and obtained the boundary conditions by the approach used by Lighthill (1952) in his spherical model for squirming motion at low Reynolds numbers. Comparing the velocities of propulsion for the infinite models (planar and cylindrical) with that for finite spherical model, studied by himself in a previous paper, Blake observed that planeness is more important than finiteness. Blake's analysis provides some support to infinite oscillating (waving) sheet model.

All of the above references are concerned with Newtonian liquids. However, the departure from the behaviour predicted by a Newtonian liquid has gained importance in many industrial processes, particularly in polymer industries, in recent years. Literature now abounds in studies of the response of non-Newtonian liquids to various flow situations. Chang & Schowalter (1974, 1975) reported that patterns of secondary flows induced by an oscillating cylinder were drastically altered when a small amount of polymers was added to a Newtonian liquid. Chang (1977) attempted to explain this phenomenon qualitatively by using unsteady boundary-layer equations with Walters' liquid B' (1964) model for a viscoelastic liquid. This oscillating cylinder was however not a flexible waving cylinder, as considered by Blake, but a cylinder oscillating as a rigid body. Dandapat & Gupta (1975) theoretically studied the instability of a horizontal layer of a viscoelastic liquid on an oscillating plane. The viscoelastic liquid model used by them was 'incompressible second-order fluid' of Coleman & Noll (1960). Their oscillating plane also was a rigid boundary and not a flexible waving sheet as considered by Taylor or Blake. The novel feature of the result obtained by Dandapat & Gupta is that the role of elastic property of the liquid is destabilizing in a certain frequency range and stabilizing in some other frequency range.

Biological fluids also are believed to be non-Newtonian in character. In the biological world many important functions are performed by ciliary motion. Examples of some of these are transport of gametes in the reproductive system, transport of fluid for such tasks as feeding, respiration and excretion. It may be noted here that Taylor's (1951) analysis was specifically addressed to the problem of self-propelling of a bull's spermatozoon in bull's semen. Although the study of the motion of non-Newtonian fluids has received a great deal of attention in recent years and there is a growing interest in the problems of ciliary motion (propulsion of ciliated organisms or movement of fluids and particles through pipes whose walls are covered by cilia), to the best of the author's knowledge, no attempt has so far been made to study ciliary motion in a non-Newtonian fluid.

In this paper we consider the self-propulsion of an infinite flexible sheet executing transverse waving oscillations in an 'incompressible second-order fluid' (Coleman & Noll 1960), and attempt to study the effect of the elastic property of the fluid on the propulsion-velocity. This extends Taylor's and Tuck's problems to a class of non-Newtonian (viscoelastic) liquids. Tuck's elegant formulation and analysis have been closely followed. In view of the importance of non-Newtonian liquids in biology as well as in chemical industries, the present study is considered valuable. The second-order fluid model is, of course, valid only as a limit of more general models and can predict only trends away from the Newtonian result. But the fact that it is a second-order approximation to a variety of models of non-Newtonian fluid (the first-order approximation being Newtonian fluid) indicates that the non-Newtonian trend predicted by this model may be applicable to a wide class of non-Newtonian liquids. Moreover, at the point of applying a more general model to specific cases, one either makes simplifying assumptions to formulate the specific problem and thus reduces the fluid essentially to a second-order fluid (as in Chang's 1977 paper), or one finds the results so truncated as to be equivalent to a second-order (or equally confining) fluid model. So, one can reasonably start with a second-order fluid model to investigate the trend of non-Newtonian behaviour in a specific problem.

2. Formulation and analysis

Using the postulate of gradually fading memory to the memory functional occurring in the constitutive equation of an incompressible 'simple fluid', Coleman & Noll (1960) defined an 'incompressible second-order fluid' by the constitutive equation

$$\tau_{ij} = -p\delta_{ij} + \eta_0 A_{(1)ij} + \beta_0 A_{(1)ik} A_{(1)kj} + \nu_0 A_{(2)ij}, \tag{2}$$

where τ_{ij} is the stress-tensor, p is an indeterminate pressure and η_0, β_0, ν_0 are material constants, known as viscosity, cross-viscosity and viscoelasticity coefficients respectively, and $\nu_0 < 0$ from thermodynamic considerations. The rate-of-strain tensor $A_{(1)ij}$ and the acceleration tensor $A_{(2)ij}$ are defined by

$$A_{(1)ij} = v_{i,j} + v_{j,i},$$
$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i} v_{m,j},$$

where v_i's are the velocity components and a_i's are the acceleration components given by

$$a_i = \frac{\partial v_i}{\partial t} + v_j v_{i,j}.$$

This model (2) is really an approximation of order 2 for simple fluids. It exhibits normal stress effects which are generally observed in flows of dilute polymer solutions and it is an internally consistent approximation to the stress-relaxing fluid due to Oldroyd (1950), if relaxation time of the fluid is sufficiently small compared with the time scale of the motion. For two-dimensional motion of a fluid with constitutive equation (2), it can be easily seen that shear stress component τ_{xy} becomes independent of the cross-viscosity β_0 by virtue of the equation of continuity. Normal stress components, however, depend on β_0. The two component equations of motion involve β_0; but elimination of p between these two equations gives rise to an equation independent of β_0 again by virtue of the equation of continuity. Thus in two-dimensional flow of an incompressible second-order fluid cross-viscosity does not affect the velocity field though it modifies the pressure field. We introduce the stream function ψ for two-dimensional motion, so that the velocity components (u, v) are given by

$$u = \psi_y, \quad v = -\psi_x. \tag{3}$$

Then the differential equation for ψ is obtained as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{\nu_0}{\rho}\nabla^4\psi\right) - \frac{\eta_0}{\rho}\nabla^4\psi = \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} - \frac{\nu_0}{\rho}\frac{\partial(\psi, \nabla^4\psi)}{\partial(x,y)}, \tag{4}$$

where ρ is the density of the fluid. This equation (4) determining the velocity field is independent of β_0 as stated above; and the corresponding equation for Newtonian fluid can be recovered from it by setting $\nu_0 = 0$.

Therefore, our field equation is (4) and the boundary conditions are (as in Taylor 1951)

$$\left.\begin{aligned} u &= \tfrac{1}{4}b^2\kappa\sigma\cos(2\kappa x - 2\sigma t) + O(b^4), \\ v &= -b\sigma\cos(\kappa x - \sigma t) + O(b^3), \end{aligned}\right\} \tag{5}$$

on the moving surface $y = b\sin(\kappa x - \sigma t)$. Following Tuck's (1968) notation and analysis, we assume the expansion

$$\psi = \mathrm{Re}\,[\psi_1(y)\exp\{-i(\kappa x - \sigma t)\}] + \Psi_2(y) + \mathrm{Re}\,[\psi_2(y)\exp\{-2i(\kappa x - \sigma t)\}] + O(b^3), \tag{6}$$

where the first term in (6) is $O(b)$ and satisfies the linearized version of the equation (4), while the remaining second-order terms are divided into a non-harmonic part $\Psi_2(y) = O(b^2)$ independent of t and x, and a second-harmonic part which varies sinusoidally in t and x. This second-harmonic part will not be required for our purpose here. The non-harmonic part has got to be introduced to satisfy the second-order boundary conditions.

For the linearized flow given by the differential equation

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{\nu_0}{\rho}\nabla^4\psi\right) - \frac{\eta_0}{\rho}\nabla^4\psi = 0,$$

the solution in the form assumed in (6) with appropriate boundary conditions (5) [first-order boundary conditions] is obtained as

$$\psi_1 = -\left(\frac{\eta_0}{\rho} + \frac{i\sigma\nu_0}{\rho}\right) bl(l+\kappa)\left[\frac{e^{-ly}}{l} - \frac{e^{-\kappa y}}{\kappa}\right], \tag{7}$$

where
$$l = \left\{\left(\kappa^2 + \frac{\rho\sigma^2\nu_0}{\eta_0^2 + \sigma^2\nu_0^2}\right) + \frac{i\rho\sigma\eta_0}{\eta_0^2 + \sigma^2\nu_0^2}\right\}^{\frac{1}{2}}.$$

The equation satisfied by Ψ_2, the non-harmonic part of the second approximation, is

$$-\frac{\eta_0}{\rho}\frac{d^4\Psi_2}{dy^4} = \left\langle\frac{\partial(\psi_1, \nabla^2\psi_1)}{\partial(x,y)}\right\rangle - \frac{\nu_0}{\rho}\left\langle\frac{\partial(\psi_1, \nabla^4\psi_1)}{\partial(x,y)}\right\rangle,$$

where the angle brackets denote an average with respect to x or t;

$$-\frac{\eta_0}{\rho}\frac{d^4\Psi_2}{dy^4} = \tfrac{1}{2}\kappa\left(1 + \frac{2\nu_0\kappa^2}{\rho}\right)\frac{d}{dy}\left\{\mathrm{Re}\left(i\bar{\psi}_1\frac{d^2\psi_1}{dy^2}\right)\right\} - \frac{1}{2}\cdot\frac{\kappa\nu_0}{\rho}\frac{d}{dy}\left\{\mathrm{Re}\left(i\bar{\psi}_1\frac{d^4\psi_1}{dy^4}\right)\right\};$$

the overbar denotes the complex conjugate. The solution for Ψ_2 which corresponds to a velocity U_∞ at $y = \infty$ is

$$\Psi_2 = U_\infty y + \tfrac{1}{2}b^2\sigma \cdot \frac{1-\sigma^2\nu_0^2/\eta_0^2}{1+\sigma^2\nu_0^2/\eta_0^2}|\alpha|^2 \operatorname{Re}\left[\frac{l}{\alpha^3}e^{-\alpha y} - \frac{\kappa}{\gamma^3}e^{-\gamma y}\right] - b^2\sigma \cdot \frac{\sigma\nu_0/\eta_0}{1+\sigma^2\nu_0^2/\eta_0^2}|\alpha|^2 \operatorname{Im}\left(\frac{l}{\alpha^3}e^{-\alpha y}\right), \quad (8)$$

where $\alpha = \kappa + \bar{l}$, $\gamma = l + \bar{l} = 2\operatorname{Re}(l)$.

The boundary condition to be satisfied on $y = 0$ is obtained by the substitution of the expansion (6) into the boundary conditions (5), resulting in

$$\frac{d\Psi_2}{dy} = -\left\langle \operatorname{Re}\left[\frac{d\psi_1}{dy}e^{-i(\kappa x - \sigma t)}\right]_{y = b\sin(\kappa x - \sigma t)}\right\rangle = \tfrac{1}{4}\sigma b^2\gamma. \quad (9)$$

Therefore, from (8) we must have

$$U_\infty = \tfrac{1}{4}\sigma b^2\gamma + \tfrac{1}{2}b^2\sigma \cdot \frac{1-\sigma^2\nu_0^2/\eta_0^2}{1+\sigma^2\nu_0^2/\eta_0^2}|\alpha|^2 \operatorname{Re}\left[\frac{l}{\alpha^2} - \frac{\kappa}{\gamma^2}\right] - b^2\sigma\frac{\sigma\nu_0/\eta_0}{1+\sigma^2\nu_0^2/\eta_0^2}|\alpha|^2 \operatorname{Im}\left(\frac{l}{\alpha^2}\right).$$

After some involved calculations, we eventually find that

$$U_\infty = c \cdot \tfrac{1}{2}(\kappa b)^2\left[\frac{f+1}{2f} + \frac{\lambda^2}{1+\lambda^2}\frac{2f^2-f-1}{f} - \frac{1-\lambda^2}{(1+\lambda^2)^2}\frac{R\lambda\{2f(f+1)(4f+3)(1+\lambda^2)+R\lambda\}}{4f^2\{2f(f+1)(1+\lambda^2)+R\lambda\}} + \frac{2\lambda}{(1+\lambda^2)^{\frac{3}{2}}}\frac{2(f+1)^2(1+\lambda^2)^2 - R^2\lambda^2}{2f(f+1)(1+\lambda^2)+R\lambda}\{(1+\lambda^2)(f^2-1)+R\lambda\}^{\frac{1}{2}}\right] \quad (10)$$

to second order in $(b\kappa)$, where $c = \sigma/\kappa$ is the phase velocity of the wave of displacement, $R = \rho\sigma/\eta_0\kappa^2$ is the Reynolds number based on the phase velocity, $M = -\nu_0\kappa^2/\rho$ is the elastic parameter, $\lambda = RM = -\sigma\nu_0/\eta_0$, and

$$f(R,\lambda) = \frac{\gamma}{2\kappa} = \frac{1}{\kappa}\operatorname{Re}(l) = \operatorname{Re}\left[\left\{\left(1 - \frac{R\lambda}{1+\lambda^2}\right) + \frac{iR}{1+\lambda^2}\right\}^{\frac{1}{2}}\right] = \left[\frac{(1+\lambda^2 - R\lambda) + \{(1+R^2) - \lambda(R-\lambda)(2+\lambda^2-R\lambda)\}^{\frac{1}{2}}}{2(1+\lambda^2)}\right]^{\frac{1}{2}}. \quad (11)$$

When $M = 0$ and therefore $\lambda = 0$ (i.e. the fluid is not visco-elastic but only viscous)

$$f(R,0) = \left[\frac{1+(1+R^2)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}} = F(R),$$

which is Tuck's expression (1.3); and then $U_\infty = c \cdot \frac{1}{2}(\kappa b)^2 \cdot [F(R)+1]/2F(R)$, which is Tuck's result (1.4). When $R \to 0$ and therefore $\lambda \to 0$ also, then $f(R,\lambda) \to 1$ and therefore $U_\infty = c \cdot \frac{1}{2}(\kappa b)^2$, which is Taylor's result.

Now for the validity of the second-order fluid model as a consistent constitutive approximation the frequency of oscillation must not be too large and also we must take $M \ll 1$ and $\lambda \equiv RM \ll 1$ (Denn, Sun & Rushton 1971; Porteous & Denn 1972). Here we take $M = O(10^{-3})$. We write $\bar{U}_\infty = U_\infty/\frac{1}{2}c(\kappa b)^2$. Figures 1 and 2 show the variation of $\bar{U}_\infty$ with R for several values of M and of λ respectively. Figure 1 clearly shows that the elastic property of the fluid augments self-propulsion in some range of the Reynolds number R $(0 < R < R_1^*)$ and hampers it in some other range $(R > R_2^*)$.

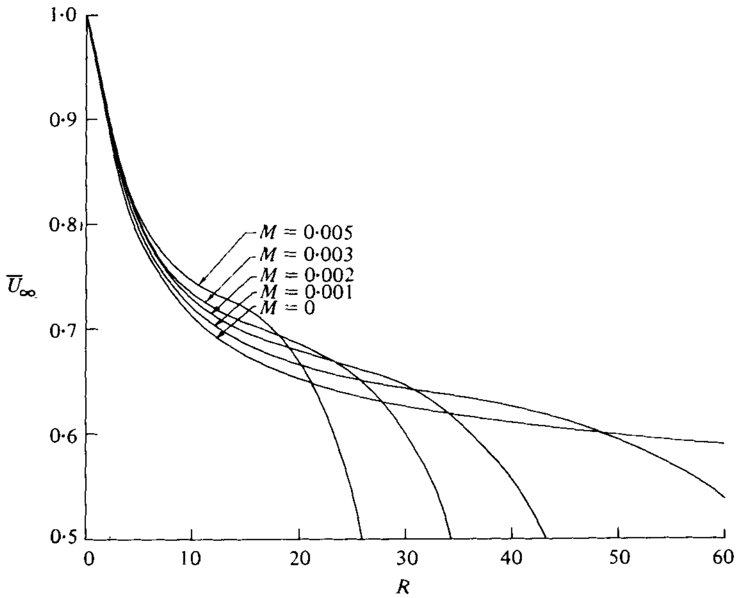

FIGURE 1. The variation of $\bar{U}_\infty$ with R for several values of M.

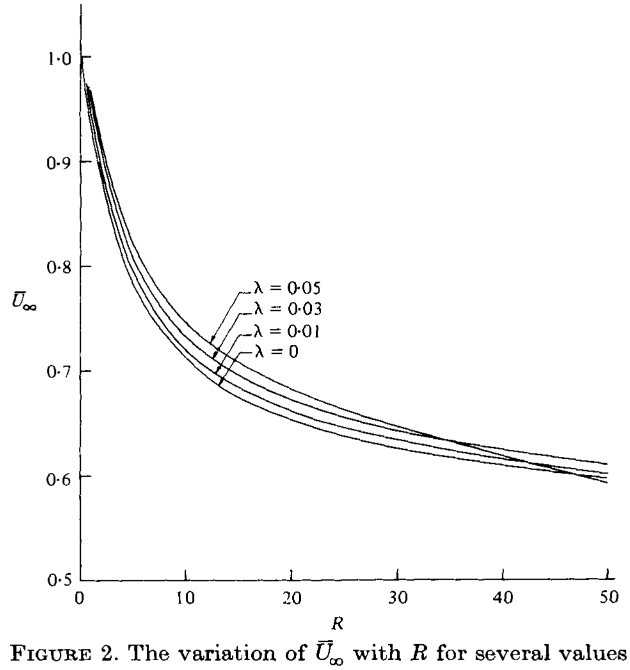

FIGURE 2. The variation of $\bar{U}_\infty$ with R for several values of λ.

For each M, the increase in self-propulsion owing to elasticity of the fluid increases with increasing R up to $R = R_1^*$, then this increase falls sharply to zero at $R = R_2^*$ (where the corresponding curve crosses the curve for $M = 0$), and for larger R the self-propulsion is decreased very sharply. Critical values R_1^* and R_2^* decrease as M increases. For $M = 0{\cdot}001$, $R_1^* \simeq 34$ and $R_2^* \simeq 49$; and for $M = 0{\cdot}005$, $R_1^* \simeq 15$

and $R_2^* \simeq 21$. The maximum increase in self-propulsion due to elasticity is nearly 3 % in the first case and 6 % in the second case. It may be noted here that, for $M = 0{\cdot}001$, the elasticity begins to hamper propulsion at $\lambda \simeq 0{\cdot}049$, for which $\lambda \ll 1$ is fulfilled. For $M = 0{\cdot}005$ the corresponding value of λ is $0{\cdot}105$, which is however not that small. It may be of some interest to look into the contribution of different terms of (10) towards the peculiar nature of the variation of U_∞ with R. From computations we find that the magnitudes of all the terms increase with M and for a fixed M the first term decreases with R while all other terms increase with R (within the range of R as shown in the graphs) from zero at $R = 0$. A study of the relative order of the magnitudes of the terms reveals that the crucial role is played by the third term of (10) which comes partly from the nonlinear inertial term $\partial(\psi, \nabla^2\psi)/\partial(x, y)$ through the ν_0-term of ψ_1, the linearized flow solution, and partly from the nonlinear visco-elastic term $(\nu_0/\rho)\,\partial(\psi, \nabla^4\psi)/\partial(x, y)$.

The time-averaged rate of dissipation of energy per unit area of the sheet in the whole fluid (fluid on both sides of the sheet) is given by (cf. Taylor 1951)

$$\langle \dot{E} \rangle = \left\langle \left[-2\frac{dy}{dt}\tau_{yy} \right]_{y = b\sin(\kappa x - \sigma t)} \right\rangle.$$

The normal stress τ_{yy} involves the cross-viscosity coefficient β_0 and therefore, in general, $\langle \dot{E} \rangle$ depends on cross-viscosity. But the estimate of this rate of dissipation to second order in b (the present analysis is only concerned with terms up to second order in b) is independent of β_0 and takes a particularly simple form. We get

$$\langle \dot{E} \rangle = \eta_0\sigma^2 b^2\kappa \left[(1+f) + \frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}}\{(1+\lambda^2)(f^2-1) + R\lambda\}^{\frac{1}{2}} \right]$$

to second order in b.

This $\langle \dot{E} \rangle$ reduces to Taylor's result $2\eta_0\sigma^2 b^2\kappa$ at $R = 0$ and to Tuck's result

$$\eta_0\sigma^2 b^2\kappa[1 + F(R)]$$

at $M = 0$. It is found to increase steadily with M as well as with R.

3. Discussion

The result obtained in this study may not be of much value quantitatively owing to various approximations involved, but we believe that it is of much significance with regard to the qualitative effect of the elastic property of the fluid on self-propulsion and particularly the reversal of the role of elasticity in different ranges of Reynolds number.

The novel effect of visco-elasticity predicted by our analysis is not entirely surprising. Chang & Schowalter (1974) speculate that the experimentally observed drastic change of secondary flow results from the elasticity of the fluid (dilute Separan solution). Chang (1977), from his theoretical analysis, finds that the effect of elasticity of the fluid is to greatly increase the secondary flow within the boundary layer. Dandapat & Gupta's (1975) stability analysis reveals a novel feature of reversal of the role of viscoelasticity (from a destabilizing agent for low frequencies to a stabilizing agent for high frequencies) quite similar to that revealed by our present analysis. Supported by experimental evidence they anticipate that the unusual effect (stabilizing) of elasticity is manifested when the flow is of the boundary-layer type. In our analysis elasticity of the fluid also reverses its role and hampers propulsion (reduces induced velocity) when the Reynolds number is sufficiently increased, that is, the flow is of the boundary-layer type. Thus the present analysis leads us to believe more strongly that the influence of elasticity must be quite different for boundary-layer flows. We have found from computed values that, when R is sufficiently increased, the elastic effect is dictated by a part of the nonlinear terms. We, therefore, feel that precise identification (not attempted at present) of the origin of this part may help in understanding the apparently unusual influence of elasticity on boundary-layer-type flows. An approach analogous to that initiated recently by Gatski & Lumley (1978) of numerically solving simultaneously the motion and stress-field equations with a suitable constitutive equation, will probably be more appropriate for obtaining significant information.

We end our discussion with a few words on the infinite-sheet model for self-propelling organisms. As has been already mentioned in the introduction, Blake's (1971) investigation dispels some of the natural doubts about this model. For the infinite-sheet model Taylor (1951) has used the concept of the propelling organ as a thin tail down which the organism sends waves of displacement, while Blake (1971) has used the concept of instantaneous surface covering the numerous cilia over the body of the organism. Blake has observed that ciliated organisms (like *Paramecium* or *Opalina*) tend to be elongated or flat so that a high ratio of surface area to volume can occur and that some multi-cellular animals have ciliated epithelia which can be considered to be effectively an infinite sheet of ciliated surface. Blake has also shown that a small amplitude perturbation (as we have done in our analysis also) is valid for *Opalina*. Further refinement in the models for ciliary motion has been introduced by Blake (1972) by means of discrete-cilia approach and is pursued by several workers. However, if fluid is to be continuously moved along the top of a (plane) cilia layer, then the rows of cilia should be very densely packed (as indeed observed in ciliated organisms) and in that case a two-dimensional flow is approached (Liron 1978). The velocity immediately above the cilia layer is uniform and almost time-independent (Liron & Mochon 1976).

We conclude by saying that as a first step to enter into the realm of ciliary motion in non-Newtonian fluids and to gain some insight into it one seems to be quite justified to consider an infinite-sheet model for the propelling body and a second-order model for non-Newtonian fluid.

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